SOME THREE-DIMENSIONAL PROBLEMS OF THERMOELASTICITY

(O NEKOTORYKH PROSTRANSTVENNYKH ZADACHAKH TERMOUPRUGOSTI)

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In this paper, thermal stresses are investigated which are caused by the action of non-stationary sources of heat, arbitrarily distributed in elastic and visco-elastic media. The construction of Green's functions for stresses caused by an instantaneous point source of heat is considered. In Sections 1 and 2 the state of stress in absolutely elastic bodies is considered and in Section 3, in visco-elastic bodies.

1. The state of stress in an infinite elastic medium. As is known from the theory of thermal conductivity, the temperature field due to the action of an instantaneous point source of heat is described by the formula

$$T = \frac{Q}{(\pi\vartheta)^{3/2}} \exp\left(-\frac{R^2}{\vartheta}\right), \qquad \vartheta = 4\varkappa t, \qquad \varkappa = \frac{\lambda}{c\rho} \tag{1.1}$$

Here $W = Q\rho\, c$ is the quantity of heat received in unit volume in unit time, ρ is the density, c is the specific heat, and λ is the coefficient of thermal conductivity. The function (1.1) is the solution of the equation

$$\nabla^2 T - \frac{1}{\varkappa}\frac{\partial T}{\partial t} = -\frac{Q}{\varkappa}\delta(R)\,\delta(t) \tag{1.2}$$

$$T(R,\ t)_{t=0} = 0, \qquad T(\infty,\ t) = 0, \qquad T(R,\ \infty) = 0$$

where δ is the Dirac symbol.

We consider first the quasi-static problem. The equations of the theory of elasticity for displacements, if the inertial terms are neglected, can be represented in the form

$$\mu\nabla^2 u_i + (\lambda + \mu)\frac{\partial\theta}{\partial x_i} = (3\lambda + 2\mu)\,\alpha_t \frac{\partial T}{\partial x_i} \qquad (i = 1,\ 2,\ 3) \tag{1.3}$$

λ, μ are the Lamé constants, α_t is the coefficient of thermal expansion.

We introduce the potential of thermoelastic displacement ϕ. This potential is related to the displacement by the equation [1]

$$u_i = \frac{\partial \varphi}{\partial x_i} \qquad (i = 1, 2, 3) \tag{1.4}$$

Introducing the function ϕ into equations (1.3) we reduce the system of equations for displacement (1.3) to one equation [1]

$$\nabla^2 \varphi = \vartheta_0 T \qquad \left(\vartheta_0 = \frac{1+\nu}{1-\nu}\alpha_t\right) \tag{1.5}$$

Knowing the function ϕ, it is possible according to formulas [1] to determine the stresses

$$\sigma_{ij} = 2G\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij}\nabla^2\right)\varphi \tag{1.6}$$

where G is the shear modulus, and δ_{ij} is the Kronecker symbol.

In a finite body, the function ϕ at best satisfies only part of the boundary conditions; so that to the stresses expressed by the formula (1.6), one must add such selected stresses as to satisfy all the boundary conditions.

In the problem under consideration, we take advantage of spherical symmetry; we have

$$\frac{\partial^2 \varphi}{\partial R^2} + \frac{2}{R}\frac{\partial \varphi}{\partial R} = \vartheta_0 T, \qquad u_R = \frac{\partial \varphi}{\partial R} \tag{1.7}$$

$$\sigma_{RR} = -2G\frac{2}{R}\frac{\partial \varphi}{\partial R}, \qquad \sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = -2G\left(\frac{1}{R}\frac{\partial \varphi}{\partial R} + \frac{\partial^2 \varphi}{\partial R^2}\right) \tag{1.8}$$

The expression (1.1) for the temperature field, by use of the Laplace transform, can take the form of the following Hankel-Fourier integral

$$\theta = \frac{Q}{2\pi^2\varkappa}\int_0^\infty\int_0^\infty \alpha(\alpha^2 + \gamma^2 + p/\varkappa)^{-1} J_0(\alpha r)\cos\gamma z \, d\alpha \, d\gamma$$

$$\theta = \int_0^\infty e^{-pt} T(R, t)\, dt \tag{1.9}$$

By use of the Laplace transform, from equation (1.7), taking $Q = 1$, we have

$$\Phi^* = -\frac{\vartheta_0}{2\pi^2\varkappa}\int_0^\infty\int_0^\infty \alpha\left(\alpha^2 + \gamma^2 + \frac{p}{\varkappa}\right)^{-1}(\alpha^2 + \gamma^2)^{-1} J_0(\alpha r)\cos\gamma z \, d\alpha \, d\gamma \tag{1.10}$$

After carrying out the inverse Hankel-Fourier transform, we obtain

$$\Phi^* = \frac{\vartheta_0}{4\pi K p}\left[\exp\left(-R\sqrt{\frac{p}{\varkappa}}\right) - 1\right]$$

Hence

$$\varphi^* = -\frac{\vartheta_0}{4\pi R}\operatorname{erf}\frac{R}{\sqrt{\vartheta}}, \qquad \vartheta = 4\varkappa t, \qquad \operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\eta^2} d\eta \tag{1.11}$$

Knowing the function ϕ^*, we determine the stresses [2] as

$$\sigma^*_{RR} = -\frac{4GA}{R^3}\left[\operatorname{erf}\frac{R}{\sqrt{\vartheta}} - \frac{2R}{\sqrt{\pi\vartheta}}\exp\left(-\frac{R^2}{\vartheta}\right)\right] \tag{1.12}$$

$$\sigma^*_{\varphi\varphi} = \sigma^*_{\vartheta\vartheta} = \frac{2GA}{R^3}\left[\operatorname{erf}\frac{R}{\sqrt{\vartheta}} - \frac{2R}{\sqrt{\pi\vartheta}}\left(1 + \frac{2R^2}{\vartheta}\right)\exp\left(-\frac{R^2}{\vartheta}\right)\right] \qquad \left(A = \frac{\vartheta_0}{4\pi}\right)$$

In Fig. 1 *a*, *b*, and *c* curves of the dependence of T^*, σ^*_{RR}, $\sigma^*_{\phi\phi}$ on R are presented for several values of the parameter θ indicated on the curves.

For $R \to \infty$, at an arbitrary moment t, the stress approaches zero. Also for finite values of R, but for $t \to \infty$, the stresses σ^*_{ij} vanish.

The functions σ^*_{RR}, $\sigma^*_{\phi\phi}$, $\sigma^*_{\theta\theta}$ can be considered as Green's functions. Let $Q(P, t)$ be the intensity of the sources of heat distributed in the region Γ; then

$$\sigma_{ij}(P, t) = \iiint_{(\Gamma)}\int_0^t Q(S, t')\,\sigma^*_{ij}(S, P, t-t')\,d\Gamma dt' \tag{1.13}$$

In an analogous manner, we have

$$\varphi(P, t) = \iiint_{(\Gamma)}\int_0^t Q(S, t')\,\varphi^*(P, S, t-t')\,dt' \tag{1.14}$$

For a continuous source of heat, we obtain

$$\varphi(R, t) = \frac{AR}{2\varkappa}\left[1 - \left(1 + \frac{\vartheta}{2R^2}\right)\operatorname{erf}\frac{R}{\sqrt{\vartheta}} - \frac{1}{R}\sqrt{\frac{\vartheta}{\pi}}\exp\left(-\frac{R^2}{\vartheta}\right)\right] \tag{1.15}$$

For a source changing according to a harmonic law

$$\varphi(R, t) = \frac{Aie^{i\omega t}}{\varkappa\eta R}\left[1 - \exp\left(-R\sqrt{i\eta}\right)\right] \qquad \left(\eta = \frac{\omega}{\varkappa}\right) \tag{1.16}$$

If in the equations of motion of the theory of elasticity, the inertial terms are not neglected, then in the case considered, with spherical symmetry, along with the equations (1.7) and (1.8), we obtain the following formulas:

$$\frac{\partial^2\varphi}{\partial R^2}+\frac{2}{R}\frac{\partial\varphi}{\partial R}-\sigma^2\frac{\partial^2\varphi}{\partial t^2}=\vartheta_0 T, \qquad u_R=\frac{\partial\varphi}{\partial R} \tag{1.17}$$

$$\sigma_{RR}=-2G\frac{2}{R}\frac{\partial\varphi}{\partial R}+\rho\frac{\partial^2\varphi}{\partial t^2}, \quad \sigma_{\varphi\varphi}=\sigma_{\vartheta\vartheta}=-2G\left(\frac{\partial^2\varphi}{\partial R^2}+\frac{1}{R}\frac{\partial\varphi}{\partial R}\right)+\rho\frac{\partial^2\varphi}{\partial t^2} \tag{1.18}$$

Here
$$\sigma^2=\frac{1}{c_1^2}, \qquad c_1=\left(\frac{1-\nu}{1-2\nu}\frac{2G}{\rho}\right)^{1/2}$$

c_1 is the velocity of propagation of an elastic longitudinal wave, ρ designates the density.

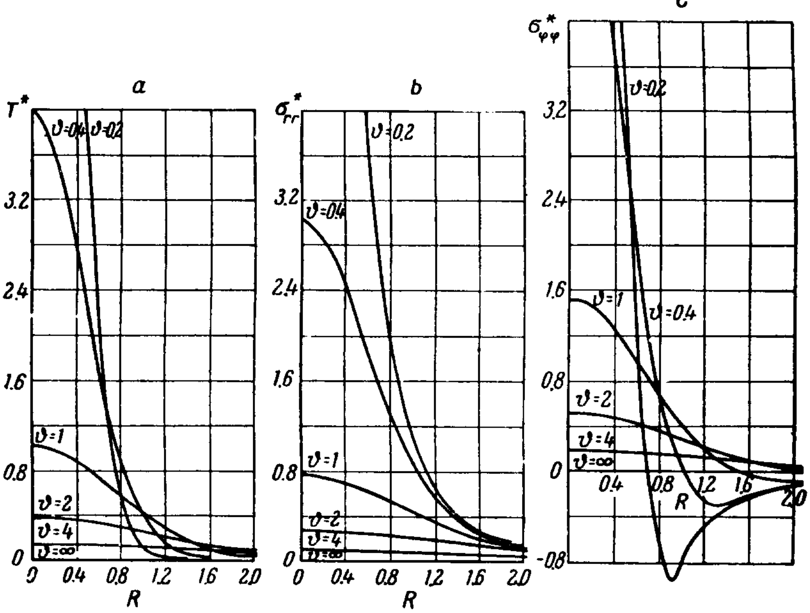

Fig. 1

Proceeding in a similar fashion to the quasi-static case, we obtain for

$$\Phi^*=-\frac{\vartheta_0}{2\pi^2\varkappa}\int_0^\infty\int_0^\infty \alpha\,(\alpha^2+\gamma^2+p/\varkappa)^{-1}(\alpha^2+\gamma^2+p^2\sigma^2)^{-1}\cos\gamma z\,d\alpha\,d\gamma=$$

$$=\frac{\vartheta_0}{4\pi\varkappa\sigma^2 pR}\,\frac{e^{-R\sigma p}-e^{-R\sqrt{p/\varkappa}}}{p-\varkappa^{-1}\sigma^{-2}}$$

Carrying out the inverse transformation, we obtain [3]

$$\varphi^*=\frac{\vartheta_0}{4\pi R}\left\{\operatorname{erfc}\frac{R}{\sqrt{\vartheta}}-\frac{1}{2}\exp\frac{\vartheta}{4\varkappa^2\sigma^2}\left[\exp\frac{R}{\varkappa\sigma}\operatorname{erfc}\left(\frac{R}{\sqrt{\vartheta}}+\frac{\sqrt{\vartheta}}{2\varkappa\sigma}\right)+\right.\right.$$

$$+\exp\left(-\frac{R}{\varkappa\sigma}\right)\operatorname{erfc}\left(\frac{R}{\sqrt{\vartheta}}-\frac{\sqrt{\vartheta}}{2\varkappa\sigma}\right)\Big]+\left(\exp\frac{\vartheta-4\varkappa R\sigma}{4x^2\sigma^2}-1\right)\eta(t-R\sigma)\Big\} \quad (1.20)$$

where η is the Heaviside function.

In the case considered, we obtain different formulas for ϕ**+ }=/ in the intervals $0 < t < R\sigma$ and $t > R\sigma$. Knowing the function ϕ^*, according to formula (1.18) we obtain the stresses σ^*_{ij}. It is easy to show that both for $R \to \infty$, and also for $t \to \infty$, the function ϕ^* vanishes; and also the stresses reduce to zero.

For $t = R\sigma$ there is a discontinuity in stress. It is evident that, considering ϕ^* as a Green's function, it is possible to determine the stresses for an arbitrary function $Q(P, t)$.

We consider the state of stress due to the action of a source of heat moving in a straight line with constant speed v. Designating by ξ_1, ξ_2, and ξ_3 stationary coordinates, we assume that the source has an intensity W changing with time, and moves in an elastic medium along the axis ξ_1.

The equation of thermal conductivity in this case has the form

$$\frac{\partial^2 T}{\partial \xi_1^2}+\frac{\partial^2 T}{\partial \xi_2^2}+\frac{\partial^2 T}{\partial \xi_3^2}-\frac{1}{\varkappa}\frac{\partial T}{\partial t}=0 \quad (1.21)$$

We choose a new coordinate system x_1, x_2, and x_3, connected with the moving source of heat, and parallel to the system ξ_1, ξ_2, ξ_3. Applying a linear transformation

$$x_1=\xi_1-vt, \qquad x_2=\xi_2, \qquad x_3=\xi_3$$

we obtain equation (1.21) in the following form:

$$\frac{\partial^2 T}{\partial x_1^2}+\frac{\partial^2 T}{\partial x_2^2}+\frac{\partial^2 T}{\partial x_3^2}+2\mu\frac{\partial T}{\partial x_1}-\frac{1}{\varkappa}\frac{\partial T}{\partial t}=0, \qquad \left(\mu=\frac{v}{2\varkappa}\right) \quad (1.22)$$

In the case of a source having constant intensity, we have $\partial T/\partial t = 0$. We dwell for a while on this quasi-stationary case.

It is known that in this case we have

$$T=\frac{Q}{2\pi^2\varkappa R}e^{-\mu(x_1+R)} \qquad (R=(x_1^2+x_2^2+x_3^2)^{1/2},\ W=Q\rho c) \quad (1.23)$$

Solving the equation

$$\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}+\frac{\partial^2\varphi}{\partial x_3^2}=\vartheta_0 T \quad (1.24)$$

we obtain, for $Q = 1$

$$\varphi^* = \frac{\vartheta_0}{8\pi\mu\varkappa}\{\mathrm{Ei}[-\mu(x_1+R)] - \ln(x_1+R)\} \tag{1.25}$$

$$\mathrm{Ei}(-s) = \int_s^\infty \frac{e^{-u}}{u}\,du \quad (s>0)$$

Now it is possible, according to formula (1.6), to determine the stresses σ^*_{ij}, carrying out the appropriate differentiations of the function ϕ^*

$$\sigma_{12}^* = \frac{x_2 K}{\mu R^3}\left[(1+\mu R)e^{-\mu(x_1+R)} - 1\right]$$

$$\sigma_{23}^* = \frac{x_2 x_3 K}{\mu R^3 (x_1+R)}\left[\left(1+\mu R+\frac{R}{x_1+R}\right)e^{-\mu(x_1+R)} - \left(1+\frac{R}{x_1+R}\right)\right] \tag{1.26}$$

$$\left(K = \frac{1+\nu}{1-\nu}\,\frac{d}{8\pi\mu\varkappa}\right) \qquad \text{and so forth}$$

Carrying out in the formulas for stress the passage to the limit $\mu \to 0$ $(v \to 0)$, we obtain the well-known formulas for stress due to a stationary constant source of heat,

$$\sigma_{12}^* = -\frac{x_1 x_2 K}{R^3}, \qquad \sigma_{23}^* = -\frac{x_2 x_3 K}{R^3} \qquad \text{and so forth}$$

2. Thermal stresses in an elastic half-space. We consider an elastic half-space $(x_3 > 0)$, in which an instantaneous source of heat acts at the point $(0, 0, \xi_3 = \zeta)$. We assume that the plane $x_3 = 0$ is free of stress. Further, we assume that $T = 0$ for $x_3 = 0$. Here the problem is axially-symmetrical, so that cylindrical coordinates can be used to describe the system, namely, r and z. The first two boundary conditions

$$\sigma_{zz}^* = 0, \qquad T = 0, \qquad \sigma_{rz}^* = 0 \qquad \text{for } z = 0 \tag{2.1}$$

are satisfied, if in the unbounded elastic space we place at the point $(0, \zeta)$ a positive, and at the point $(0, -\zeta)$ a negative source of heat. Then, in conformity with formula (1.11), we obtain [4]

$$\varphi^*(r, z) = -A\left[\frac{1}{R_1}\operatorname{erf}\left(\frac{R_1}{\sqrt{\vartheta}}\right) - \frac{1}{R_2}\operatorname{erf}\left(\frac{R_2}{\sqrt{\vartheta}}\right)\right]$$

$$R_{1,2} = [r^2 + (z \mp \zeta)^2]^{1/2}, \qquad A = \frac{\vartheta_0}{4\pi} \tag{2.2}$$

As is known, the stresses are expressed through the function ϕ^* in the following manner:

$$\sigma_{rr}^{*\prime} = -2G\left(\frac{1}{r}\frac{\partial\varphi^*}{\partial r} + \frac{\partial^2\varphi^*}{\partial z^2}\right), \qquad \sigma_{\varphi\varphi}^{*\prime} = -2G\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{\partial^2\varphi^*}{\partial z^2}\right)$$

$$\sigma_{zz}^{*\prime} = -2G\left(\frac{1}{r}\frac{\partial\varphi^*}{\partial r} + \frac{\partial^2\varphi^*}{\partial v^2}\right), \qquad \sigma_{rz}^{*\prime} = 2G\frac{\partial^2\varphi^*}{\partial r\partial z} \tag{2.3}$$

In the plane $z = 0$, the stress $\sigma^{*\prime}_{rz}$ does not vanish. In order that the stress $\sigma^{*\prime}_{rz}(r, 0, t)$ should reduce to zro, we impose such a state of stress that in the plane $z = 0$, the condition should be satisfied

$$\sigma_{rz}{}^{*\prime}(r, 0, t) + \sigma^{*\prime\prime}{}_{rz}(r, 0, t) = 0, \qquad \sigma_{zz}{}^{*\prime\prime}(r, 0, t) = 0 \quad (2.4)$$

We determine the state of stress $\sigma_{ij}{}^{*\prime\prime}$ by means of the Love function ϕ^0, satisfying the equation $\cdot = 0$.

The function ϕ^0 is taken in the form

$$\varphi^0 = \int_0^\infty (C + D\alpha z)\, e^{-\alpha z} J_0(\alpha v)\, d\alpha \quad \text{for } z > 0 \qquad (2.5)$$

From the second condition (2.4) it follows that $C = -(1 - 2\nu) D$. Taking into account that

$$\sigma_{rz}{}^{*\prime}(r, 0, t) = \frac{G\vartheta_0}{2\pi} \int_0^\infty \rho(\alpha, \zeta, t)\, \alpha^2 J_1(\alpha r)\, d\alpha$$

where

$$\rho(\alpha, \zeta, t) = e^{-\alpha\zeta} \operatorname{erfc}\left(\frac{\alpha\sqrt{\vartheta}}{2} - \frac{\zeta}{\sqrt{\vartheta}}\right) - e^{\alpha\zeta} \operatorname{erfc}\left(\frac{\alpha\sqrt{\vartheta}}{2} + \frac{\zeta}{\sqrt{\vartheta}}\right)$$

we obtain from the first condition (2.4)

$$D(\alpha, \zeta, t) = \frac{1 - 2\nu}{4\pi\alpha} \vartheta_0 \rho(\alpha, \zeta, t)$$

In this manner the function ϕ^0 is determined, and from it, the state of stress $\sigma_{ij}{}^{*}$, since

$$(2.6)$$

$$\sigma_{rr}{}^{*\prime\prime} = \frac{2G}{1 - 2\nu} \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\varphi^0, \qquad \sigma_{zz}{}^{*\prime\prime} = \frac{2G}{1 - 2\nu} \frac{\partial}{\partial z}\left[\left(2 - \nu\right)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi^0$$

$$\sigma_{\varphi\varphi}{}^{*\prime\prime} = \frac{2G}{1 - 2\nu} \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right)\varphi^0, \qquad \sigma_{rz}{}^{*\prime\prime} = \frac{2G}{1 - 2\nu} \frac{\partial}{\partial r}\left[(1 - \nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi^0$$

In this manner

$$\sigma_{rr}{}^{*\prime\prime} = GA \int_0^\infty \rho(\alpha, \zeta, t)\, \alpha^2 e^{-\alpha z} \left[(2 - \alpha z) J_0(\alpha r) + (2\nu - 2 + \alpha z) \frac{J_1(\alpha r)}{\alpha r}\right] d\alpha \text{ and so forth}$$

If one assumes that $\partial T/\partial z = 0$ in the plane $z = 0$, then the stresses $\sigma_{ij}{}^{*\prime}$ may be determined by placing instantaneous sources of heat located at the points $(0, \zeta)$ and $(0, -\zeta)$. In this way, in the plane $z = 0$ the conditions are satisfied $\sigma_{rz}{}^{*\prime} = 0$ and $\partial T/\partial z = 0$.

We eliminate the stress $\sigma_{zz}{}^{*\prime}(r, 0, t)$ by adding to it the states of stress $\sigma_{ij}{}^{*}$ and $\sigma_{ij}{}^{*\prime\prime}$, expressed by use of the Love function, whereby in formula (2.5) $C = 2\nu D$ should be taken.

If ζ approaches zero, we have the case of a source acting at the origin of the coordinate system, that is, in the plane bounding the elastic half-space. In this special case we obtain for a continuous source of heat

$$D(\alpha,\ t) = \frac{A}{2\varkappa\alpha^3}(1-2\nu)\left[1-\alpha\exp(-\alpha^2\varkappa t)\sqrt{\frac{\vartheta}{\pi}}-\left(1-\frac{\alpha^2}{2}\vartheta\right)\operatorname{erfc}\left(\frac{\alpha\sqrt{\vartheta}}{2}\right)\right]$$
$$(\vartheta = 4\varkappa t)$$

For a steady source of heat, that is, for $t \to \infty$ we obtain

$$D(\alpha,\ t) = \frac{A}{2\varkappa}(1-2\nu)\alpha^{-3}$$

We notice that for $t \to \infty$ the stresses $\sigma_{rz}{}^*(r,\ z,\ \infty)$ and $\sigma_{zz}{}^*(r,\ z,\ \infty)$ are equal to zero.

The stresses $\sigma_{ij}{}^*(r,\ z,\ t)$ for a continuous source of heat may be represented as

$$\sigma_{ij}{}^* = \sigma_{ij}{}^{*(0)} - \sigma_{ij}{}^{*(1)}(r,\ z,\ t)$$

where the stresses $\sigma_{ij}{}^{*(0)}(\infty)$ do not depend on time. For the stresses $\sigma_{zz}{}^*$, and $\sigma_{rz}{}^*$ we obtain

$$\sigma_{rz}{}^* = -\sigma_{rz}{}^{*(1)}(r,\ z,\ t), \qquad \sigma_{zz}{}^* = -\sigma_{zz}{}^{*(1)}(r,\ z,\ t)$$

These stresses vanish for $t \to \infty$, assuming a maximum for some finite value of t.

We consider the following problems which have importance for technical applications. Assume that in a finite region Γ, located in the plane $z = 0$ which bounds an elastic half-space, the following boundary condition for temperature is given:

$$T(x_1,\ x_2,\ 0,\ t) = f(x_1,\ x_2)\,\delta(t) \tag{2.7}$$

and let $T = 0$ on the remainder of the surface. We construct a Green's function for this problem.

The temperature field should satisfy the differential equation

$$\nabla^2 T^* - \frac{1}{\varkappa}\frac{\partial T^*}{\partial t} = 0$$

and the boundary condition

$$T^*(x_1,\ x_2,\ 0,\ t) = \delta(x_1-\xi_1)\,\delta(x_2-\xi_2)\,\delta(t) \tag{2.8}$$

as well as

$$T^* = 0 \quad \text{at infinity}$$

For the given temperature in the region Γ we obtain $(d\Gamma = d\xi_1\,d\xi_2)$

$$T(x_1, x_2, x_3, t) = \iint_\Gamma f(\xi_1, \xi_2) T^*(x_1, x_2, x_3; \xi_1, \xi_2, 0, t)\, d\Gamma$$

$$\sigma_{ij}(x_1, x_2, x_3, t) = \iint_\Gamma f(\xi_1, \xi_2)\, \sigma_{ij}^*(x_1, x_2, x_3; \xi_1, \xi_2, 0, t)\, d\Gamma \qquad (2.9)$$

We determine the Green's function for the axially-symmetric case first, solving the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) T^* - \frac{1}{\varkappa}\frac{\partial T^*}{\partial t} = 0 \qquad (2.10)$$

with the boundary condition

$$T^*(r, 0, t) = \frac{\delta(r)\,\delta(t)}{2\pi r}, \qquad T^* = 0 \quad \text{at infinity} \quad (2.11)$$

The solution of equation (2.10) is

$$T^*(r, z, t) = \frac{4\varkappa z}{\vartheta(\vartheta\pi)^{3/2}} \exp\left(-\frac{R^2}{\vartheta}\right) \qquad (\vartheta = 4\varkappa t) \qquad (2.12)$$

Knowing the function T^*, one finds ϕ^* as the solution of equation (1.5)

$$\varphi^* = -\frac{z\vartheta_0\varkappa}{2\pi R^3}\left[\operatorname{erf}\frac{R}{\sqrt{\vartheta}} - \frac{2R}{\sqrt{\pi\vartheta}}\exp\left(-\frac{R^2}{\vartheta}\right)\right] \qquad (2.13)$$

Once the function ϕ^* is known, the stresses $\sigma_{ij}^{*\prime}$ can be determined in closed form. For $z = 0$, the stress $\sigma_{zz}^{*\prime}$ vanishes; however the stress $\sigma_{rz}^{*\prime}$ is not equal to zero. Therefore it is necessary to superpose upon the state of stress $\sigma_{rz}^{*\prime}$, the state of stress $\sigma_{rz}^{*\prime\prime}$, expressed by means of the function ϕ^0, by formulas (2.6). The quantities C and D encountered in the Love function (2.5), are fixed by the boundary conditions for $z = 0$

$$\sigma_{rz}^{*\prime} + \sigma_{rz}^{*\prime\prime} = 0, \qquad \sigma_{zz}^{*\prime\prime} = 0$$

From these conditions we obtain

$$C = -(1-2\nu) D, \qquad D(\alpha, t) = (1-2\nu)\frac{\vartheta_0}{2\pi}\operatorname{erfc}\frac{\alpha\sqrt{\vartheta}}{2}$$

In the case of the temperature field satisfying equation (2.10), with the boundary conditions

$$T^*(r, 0, t) = \frac{\delta(r)}{2\pi r}\eta(t), \qquad T^* = 0 \quad \text{at infinity}$$

where the function $\eta(t)$ is the Heaviside function, we obtain

$$T^* = \frac{z}{2\pi R^3}\left[1 - \operatorname{erf}\frac{R}{\sqrt{\vartheta}} + \frac{2R}{\sqrt{\pi\vartheta}}\exp\left(-\frac{R^2}{\vartheta}\right)\right] \qquad (2.14)$$

and also

$$\varphi^* = -\frac{\vartheta_0 z}{4\pi R}\left[1-\left(1-\frac{\vartheta}{2R^2}\right)\operatorname{erfc}\frac{R}{\sqrt{\vartheta}}-\frac{1}{R}\sqrt{\frac{\vartheta}{\pi}}\exp\left(-\frac{R^2}{\vartheta}\right)\right] \quad (2.15)$$

We obtain the stresses $\sigma_{ij}{}^{*\prime}$ from formulas (1.6), and the stresses $\sigma_{ij}{}^{*\prime\prime}$ from formulas (2.6). The function ϕ^0 is determined by formula (2.5), where

$$C = -D(1-2\nu), \qquad D(\alpha,\ t) = (1-2\nu)\frac{\vartheta_0}{4\pi a^3}[1-F(\alpha,\ t)]$$

$$F(\alpha,\ t) = (1+2\alpha^2\varkappa t)\operatorname{erfc}(\alpha\sqrt{\varkappa t}) - 2\alpha\sqrt{\frac{\varkappa t}{\pi}}\exp\ (-\alpha^2\varkappa t)$$

In the special case of a stationary temperature field ($t \to \infty$), the stresses $\sigma_{rz}{}^*$ and $\sigma_{zz}{}^*$ are set equal to zero.

3. The state of stress in visco-elastic media. We consider thermal stresses caused by the action of an instantaneous source in an infinite medium, for the model of a visco-elastic body suggested by Biot [6] and Berry [7]. We extend the relations given by these authors to the case of thermal stresses. We have

$$\sigma_{ij}(x_r,\ t) = 2\int_0^t \mu(t-\tau)\frac{\partial}{\partial\tau}\varepsilon_{ij}(x_r,\ \tau)\,d\tau +$$

$$+\ \delta_{ij}\int_0^t\left\{\lambda(t-\tau)\frac{\partial\theta\ (x_r,\ \tau)}{\partial\tau} - [3\lambda(t-\tau)+2\mu(t-\tau)]\,\alpha_t\ \frac{\partial T\ (x_r,\ \tau)}{\partial\tau}\right\}d\tau \qquad (3.1)$$

The relations given apply to bodies which in the initial moment were unstressed. Let the relaxation functions be $\lambda(t)$ and $\mu(t)$, which for an absolutely elastic body reduce to the Lamé constant.

We consider first the quasi-static problem. Substituting the stress σ_{ij} into the equilibrium equations, which express the stress through displacements, and introducing the potential of thermo-visco-elastic strain ϕ by means of

$$u_i = \frac{\partial\varphi}{\partial x_i} \qquad (i = 1,\ 2,\ 3)$$

we obtain for the function ϕ, by analogy with equation (1.5), the formula

$$\int_0^t\ [2\mu(t-\tau)+\lambda(t-\tau)]\ \frac{\partial\nabla^2\varphi}{\partial\tau}\ d\tau = \alpha_t\int_0^t[3\lambda(t-\tau)+2\mu(t-\tau)]\,\frac{\partial T}{\partial\tau}\,d\tau \quad (3.2)$$

Expressing the relations (3.1) by use of the function ϕ and using equation (3.2) we obtain

$$\sigma_{ij}(x_r,\ t) = \int_0^t 2\mu(t-\tau)\frac{\partial}{\partial\tau}\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij}\nabla^2\right)\varphi(x_r,\ \tau)\,d\tau \qquad (i=1,2,3) \tag{3.3}$$

We assume that the visco-elastic body was in the initial instant free, i.e. unstressed. We carry out in equation (3.2) and the relations (3.3), the Laplace transformation,

$$\Theta(x_r,p) = \int_0^t e^{-pt}T(x_r,\ t)\,dt, \qquad \Phi(x_r,\ p) = \int_0^t e^{-pt}\varphi(x_r,\ t)\,dt$$

$$\Sigma_{ij}(x_r,\ p) = \int_0^t e^{-pt}\sigma_{ij}(x_r,\ t)\,dt$$

obtaining

$$\nabla^2\Phi(x_r,\ p) = \vartheta(p)\,\Theta(x_r,\ p) \tag{3.4}$$

and also

$$\Sigma_{ij}(x_r,\ p) = 2G(p)\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij}\nabla^2\right)\Phi(x_r,\ p) \tag{3.5}$$

The following designations are introduced:

$$\vartheta(p) = \frac{3\lambda'(p) + 2\mu'(p)}{\lambda'(p) + 2\mu'(p)}\alpha_t, \qquad G(p) = p\mu'(p)$$

We notice that for an absolutely elastic body, we have the following relationships (see formulas (1.5) and (1.6))

$$\nabla^2\Phi^\circ(x_r,\ p) = \vartheta_0\Theta(x_r,\ p) \qquad \left(\vartheta_0 = \frac{1+\nu}{1-\nu}\alpha_t\right) \tag{3.6}$$

$$\Sigma_{ij}(x_r,\ p) = 2G\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij}\nabla^2\right)\Phi^\circ\left(x_r,\ p\right) \tag{3.7}$$

where G is a constant quantity not dependent on the parameter p.

At this point we introduce the designations ϕ^0 and $\sigma_{ij}^{\ 0}$ for an absolutely elastic body.

From a comparison of (3.4) and (3.6), and also (3.5) and (3.7), it follows that

$$\Phi(x_r,\ p) = \frac{\vartheta(p)}{\vartheta_0}\Phi^\circ(x_r,\ p), \qquad \Sigma_{ij}(x_r,\ p) = \frac{G(p)\,\vartheta(p)}{G\,\vartheta_0}\Sigma_{ij}^{\ \circ}(x_r,\ p) \tag{3.8}$$

Introducing the functions $F(p)$ and $G(p)$, where**

$$F(p) = \frac{G(p)\,\vartheta(p)}{p}, \qquad H(p) = \frac{\vartheta(p)}{p} \tag{3.9}$$

** The functions $F(p)$ and $G(p)$ are assumed in such form as to insure the inverse transformation of these functions.

after an inverse Laplace transformation, from formulas (3.8) we obtain

$$\varphi(x_r, t) = \frac{1}{\vartheta_0}\int_0^t h(t-\tau)\frac{\partial}{\partial\tau}\varphi^{\circ}(x_r, \tau)\,d\tau \tag{3.10}$$

$$\sigma_{ij}(x_r, t) = \frac{1}{G\vartheta_0}\int_0^t f(t-\tau)\frac{\partial\sigma_{ij}^{\circ}}{\partial\tau}(x_r, \tau)\,d\tau$$

The expressions derived above permit the determination of the displacements and stresses in a visco-elastic body by use of solutions obtained for an absolutely elastic body. In many cases it will be more convenient to determine first the function

$$\psi(x_r, t) = \frac{1}{\vartheta_0}\int_0^t F(t-\tau)\frac{\partial}{\partial\tau}\varphi(x_r, \tau)\,d\tau \tag{3.11}$$

and to arrive with its aid at the stresses

$$\sigma_{ij}(x_r, t) = 2\left(\frac{\partial^2}{\partial x_i\,\partial x_j} - \delta_{ij}\nabla^2\right)\psi(x_r, t) \tag{3.12}$$

Let an instantaneous source of heat act at the origin of the coordinate system in a visco-elastic medium. We assume that the relaxation functions $\lambda(t)$ and $\mu(t)$ have the same relaxation time

$$\lambda(t) = \lambda_0 e^{-\varepsilon t}, \qquad \mu(t) = \mu_0 e^{-\varepsilon t}, \qquad \lambda'(p) = \frac{\lambda_0}{p+\varepsilon}, \qquad \mu'(p) = \frac{\mu_0}{p+\varepsilon} \tag{3.13}$$

Since

$$F(p) = \gamma\frac{1}{p+\varepsilon}, \qquad \gamma = \mu_0\frac{3\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0}\alpha_t$$

therefore, in conformity with formula (3.11), and taking equation (1.11) into account, we obtain

$$\psi(R, t) = -\frac{\gamma}{4\pi R}\int_0^t e^{-\varepsilon(t-\tau)}\frac{\partial}{\partial\tau}\operatorname{erf}\frac{R}{\sqrt{4\varkappa\tau}}\,d\tau = -\frac{1}{4\pi R}\left[e^{-\varepsilon t} - A(R, t)\right] \tag{3.14}$$

where

$$A(R, t) = \frac{1}{2}e^{-\varepsilon t}\left[\exp\left(-iR\sqrt{\frac{\varepsilon}{\varkappa}}\right)\operatorname{erfc}\left(\frac{R}{\sqrt{4\varkappa t}} - i\sqrt{\varepsilon t}\right) + \right.$$

$$\left. + \exp\left(iR\sqrt{\frac{\varepsilon}{\varkappa}}\right)\operatorname{erfc}\left(\frac{R}{\sqrt{4\varkappa t}} + i\sqrt{\varepsilon t}\right)\right]$$

For a continuous source we obtain

$$\psi(R, t) = -\frac{\gamma}{4\pi R\varepsilon}\left[1 - e^{-\varepsilon t} - \operatorname{erfc}\frac{R}{\sqrt{4\varkappa t}} + A(R, t)\right] \tag{3.15}$$

We obtain the stresses σ_{ij} by formula (3.12). If in the equations of equilibrium we take into account the inertial terms, then along with equation (3.4), and the relations (3.5), we obtain the following equations and relations:

$$\nabla^2\Phi(x_r, p) - p^2\sigma^2(p)\Phi(x_r, p) = \vartheta(p)\Theta(x_r, p)$$

$$\left(\sigma^2(p) = \frac{\rho}{p\,[2\mu'(p) + \lambda'(p)]}\right) \tag{3.16}$$

$$\Sigma_{ij}(x_r, p) = 2G(p)\left(\frac{\partial^2}{\partial x_i\,\partial x_j} - \delta_{ij}\nabla^2\right)\Phi(x_r, p) + \rho p^2\Phi(x_r, p) \tag{3.17}$$

We introduce the function $\Psi(x_r, p) = G(p)\,\Phi(x_r, p)$, whereupon the relation (3.17) may be given in the form

$$\Sigma_{ij}(x_r, p) = 2\left(\frac{\partial^2}{\partial x_i\,\partial x_j} - \delta_{ij}\nabla^2\right)\Psi(x_r, p) + \rho p^2\Phi(x_r, p) \tag{3.18}$$

Comparing equation (3.17) with the corresponding equation for an absolutely elastic body,

$$\nabla^2\Phi^0(x_r, p) - p^2\sigma_0^2\Phi(x_r, p) = \vartheta_0\Theta(x_r, p) \tag{3.19}$$

where σ_0^2 and θ_0 are constant quantities independent of the parameter p, it is apparent that it is impossible to construct between the functions Φ and Φ^0 such relations as were obtained in the quasi-stationary problems for an absolutely elastic body.

In the case of an instantaneous source of heat, assuming that the functions $\lambda(t)$ and $\mu(t)$ are expressed by means of the same exponential relation as before, and with the same relaxation time, ϵ^{-1}, we find that the solution of the equation (3.16) has the form

$$\Phi(R, p) = -\frac{\vartheta(p)}{4\pi\varkappa pR}\,\frac{\exp\left(-R\sqrt{p/\varkappa}\right) - \exp\left[-Rp\sigma(p)\right]}{\left(p\sigma^2(p) - \varkappa^{-1}\right)}$$

$$\left(\sigma^2(p) = \beta\,\frac{p+\varepsilon}{p},\ \beta = \frac{\rho}{\lambda_0 + 2\mu_0}\right) \tag{3.20}$$

Thus

$$\Phi(R, p) = A\left(\frac{1}{p} - \frac{1}{p-\eta}\right)\left[\exp\left(-R\sqrt{\frac{p}{\varkappa}}\right) - \exp\left[-R\sqrt{\beta p(p+\varepsilon)}\right]\right]$$

$$\left(A = \frac{\vartheta_0}{4\pi\varkappa\beta\eta R},\ \eta = \frac{1}{\varkappa\beta} - \varepsilon\right) \tag{3.21}$$

Carrying out an inverse Laplace transformation, we obtain

$$\varphi(R,t) = A\left\{\operatorname{erfc}\frac{R}{\sqrt{4\varkappa t}} - L(R,t;\eta) - N(R,t) + K(R,t;\ \varepsilon,\eta)\right\} \qquad (3.22)$$

Here the following definitions apply:

$$L(R,t;\eta) = \frac{1}{2}e^{\eta t}\left[\exp\left(-R\sqrt{\frac{\eta}{\varkappa}}\right)\operatorname{erfc}\left(\frac{R}{\sqrt{4\varkappa t}} - \sqrt{\eta t}\right) + \right.$$
$$\left. + \exp\left(R\sqrt{\frac{\eta}{\varkappa}}\right)\operatorname{erfc}\left(\frac{R}{\sqrt{4\varkappa t}} + \sqrt{\eta t}\right)\right]$$

$$N(R,t) = \left[\exp\left(-\frac{\varepsilon R\sqrt{\beta}}{2}\right) + \right.$$
$$\left. + \frac{\varepsilon R\sqrt{\beta}}{2}\int\limits_{R\sqrt{\beta}}^{t}\exp\left(-\frac{\varepsilon v}{2}\right)\frac{I_1(1/2\varepsilon\sqrt{v^2-R^2\beta})}{\sqrt{v^2-R^2\beta}}\,dv\right]\eta(t-R\sqrt{\beta})$$

$$K(R,t;\varepsilon,\eta) = \int\limits_0^t h(R,\ t-\tau)\frac{\partial}{\partial\tau}g(\tau;\ \varepsilon,\eta)\,d\tau$$

where

$$h(R,t) = \exp\left(-\frac{\varepsilon t}{2}\right)I_0(1/2\varepsilon\sqrt{t^2-R^2\beta})\,\eta(t-R\sqrt{\beta})$$

and also

$$g(t;\varepsilon,\eta) = \int\limits_0^t\frac{1}{\sqrt{\pi(t-\tau)}}\left(\frac{e^{-\varepsilon\tau}}{\sqrt{\pi\tau}} + \sqrt{\varepsilon+\eta}\,e^{\eta\tau}\operatorname{erf}\sqrt{(\varepsilon+\eta)\tau}\right)d\tau$$

For determining the stresses σ_{ij}, one more function is necessary:

$$\psi(R,t) \qquad (\Psi(R,p) = G(p)\,\Phi(R,p))$$

The function $\psi\ (R,\ p)$ has the form

$$\psi(R,p) = A_1\left(\frac{1}{p+\varepsilon} - \frac{1}{p-\eta}\right)\left[\exp\left(-R\sqrt{\frac{p}{\varkappa}}\right) - \exp\left(-R\sqrt{\beta p(p+\varepsilon)}\right)\right]$$
$$\left(A_1 = \frac{\vartheta_0\mu_0}{4\pi\varkappa\beta R(\eta+\varepsilon)}\right)$$

Carrying out the inverse Laplace transformation, we obtain

$$\psi(R,\ t) = A_1[L(R,t;-\varepsilon) - L(R,t;\eta) + K(R,t;\varepsilon,-\varepsilon) - K(R,t;\varepsilon,\eta)] \quad (3.23)$$

We determine the stresses by use of the formula

$$\sigma_{RR} = -\frac{4}{R}\frac{\partial\psi}{\partial R} + \rho\frac{\partial^2\varphi}{\partial t^2}$$

$$\sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = -2\left(\frac{\partial^2\psi}{\partial R^2} + \frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \rho\frac{\partial^2\varphi}{\partial t^2}$$

Determination of the quasi-static thermal stresses in a visco-elastic half-space introduces, in principle, no great difficulties. First the stresses σ_{ij}' are determined for an infinite medium, as was demonstrated in Section 2, and then the boundary conditions at the plane $z = 0$ are satisfied by superposition of the state of stress σ_{ij}''. The stresses $\Sigma_{ij}(x_r, p)$ can be expressed by use of the function ϕ^0, whereupon we obtain the formulas for $\Sigma_{ij}(x_r, p)$ from equations (2.6), in which in place of γ we set $\lambda'(p)/2[\lambda'(p) + \mu'(p)]$, and in place of G, we use the quantity $\mu'(p)$.

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